Tutorial on Nonlinear Modal Analysis of Mechanical Systems

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Belgium?
University of Liège?

Founded in 1817 (Belgium was not existing yet!)

All disciplines covered

17000 undergraduate students

2000 graduate students

500 faculty members.
Aerospace and Mechanical Engineering Dept.

Facts and figures:
25 Professors
85 researchers
15 staff members
5 Masters, among which the Master of Aerospace Engineering
Space Structures & Systems Laboratory (S3L)

Nonlinear dynamics
System ID & modal analysis
Constructive utilization of NL
Bifurcation analysis & management

Spacecraft structures

Orbital mechanics

Nanosatellites

Facts and figures:
1 Professor
5 Postdoc, 6 PhD students, 1 TA
Collaborations: UIUC, Torino, VUB, LMS
Financial support: ERC, ESA, FNRS BELSPO
A linear system has an output that is directly proportional to input.

Superposition principle: cornerstone of linear theory
A **nonlinear system** is a system that is not linear.

Most physical systems are inherently nonlinear in nature! (Navier-Stokes, planetary motion, rigid body rotation, etc.)
Why Care About Nonlinearity in Structural Dynamics?

1. Nonlinear dynamics is fun!
2. Nonlinear dynamics is complex.
3. There are industrial needs.

1. Meet escalating performance.
2. Nonlinearities are pervasive.
3. Nonlinearity may reveal damage.
Two Examples in Aerospace Engineering

Morane-Saulnier MS760
Collaborators:
C. Stephan @ ONERA,
P. Lubrina @ ONERA.

SmallSat spacecraft
Collaborators:
J.B. Vergniaud @ EADS-Astrium,
B. Peeters @ LMS,
A. Newerla @ ESA.
Outline of the Presentation

Sources of nonlinearity
(with real-life examples)

Fundamental differences between linear & nonlinear dynamics

What about nonlinear modes?
### Linear Systems in Structural Dynamics

5 main assumptions for \( M\ddot{x}(t) + C\dot{x}(t) + Kx(t) = f(t) \)

<table>
<thead>
<tr>
<th>Assumption</th>
<th>Consequence</th>
</tr>
</thead>
<tbody>
<tr>
<td>Small vibrations around an equilibrium</td>
<td>\rightarrow nonlinear boundary conditions</td>
</tr>
<tr>
<td>Linear elasticity</td>
<td>\rightarrow nonlinear materials</td>
</tr>
<tr>
<td>Viscous damping</td>
<td>\rightarrow nonlinear damping mechanisms</td>
</tr>
<tr>
<td>Small displ. and rotations</td>
<td>\rightarrow geometric nonlinearity</td>
</tr>
<tr>
<td>Mechanical vibrations</td>
<td>\rightarrow inherently nonlinear forces in multiphysics applications</td>
</tr>
</tbody>
</table>
Nonlinear Materials: Elastomers, Foams, Rubber

Impact of elastomeric engine mounts on A-400 M engine roll mode

J.R. Ahlquist et al., IMAC 2010
Airbus Military

Decrease in stiffness

Modal Frequency (f) and Modal Damping (d) vs. Maximum Displacement at Resonance (u)
Geometric NL: Large Displacements/Rotations

\[ \varepsilon_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \frac{\partial u_m}{\partial x_i} \frac{\partial u_m}{\partial x_j} \right) \]

Cantilever beam with a thin beam

Increase in stiffness

Force

Rel. displ.

Thin beam activated in large displacements
Combined Nonlinearity: Nonlinear BCs + Friction

Paris aircraft @ ONERA (France)

Bolted connections between wing tip and external fuel tank.

Decrease in stiffness

FRF (dB)

Low excitation level
High excitation level
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What about nonlinear modes?
<table>
<thead>
<tr>
<th>Linear</th>
<th>Nonlinear</th>
</tr>
</thead>
<tbody>
<tr>
<td>Superposition principle</td>
<td><strong>NO !</strong> By definition …</td>
</tr>
<tr>
<td><img src="x2" alt="Diagram" /> <img src="x2" alt="Diagram" /></td>
<td></td>
</tr>
<tr>
<td>Uniqueness of the solutions</td>
<td><strong>NO !</strong> Steady state response depends on transient response</td>
</tr>
<tr>
<td><img src="sinusoidal_wave" alt="Waveform" /> <img src="sinusoidal_wave" alt="Waveform" /></td>
<td></td>
</tr>
<tr>
<td>Invariance of FRFs and of modal parameters</td>
<td><strong>NO !</strong> Freq.-energy dependence</td>
</tr>
<tr>
<td><img src="sinusoidal_wave" alt="Waveform" /> <img src="sinusoidal_wave" alt="Waveform" /></td>
<td><img src="bifurcation" alt="Bifurcation" /></td>
</tr>
<tr>
<td></td>
<td>Bifurcations, quasi-periodicity and chaos.</td>
</tr>
</tbody>
</table>
Superposition Principle (Linear Case)

\[
\ddot{x} + 0.05 \dot{x} + x = F \sin \omega t
\]

Displ. \( x \) (m)

Force \( F \) (N) with \( \omega = 1.1 \) rad/s
No Superposition Principle (Nonlinear Case)

\[ \ddot{x} + 0.05\dot{x} + x + x^3 = F\sin\omega t \]

Discontinuous behavior, potentially dangerous in practice

Displ. \( x \) (m)

Force \( F \) (N) with \( \omega = 1.1 \text{ rad/s} \)
Non-Uniqueness (Sensitivity to Initial Conditions)

\[ \ddot{x} + 0.05\dot{x} + x + x^3 = 0.15 \sin 1.6t \]

Displ. \( x \) (m)

- \( x(0) = 0, \dot{x}(0) = 0 \)
- \( x(0) = 0, \dot{x}(0) = 2 \)
Marked Sensitivity to Forcing (→ \text{Jump})

\begin{align*}
\ddot{x} + 0.05\dot{x} + x + x^3 &= 0.05 \sin 1.22t \\
\ddot{x} + 0.05\dot{x} + x + x^3 &= 0.05 \sin 1.24t
\end{align*}
How to Compute the “FRFs” of a Linear System?

\[ \ddot{x} + x = A \sin \omega t \]

\[ x(t) = B \sin \omega t \]

\[ -\omega^2 B + B = A \]

\[ B = \frac{A}{1 - \omega^2} \]

Confirmation of the superposition principle
How to Compute the “FRFs” of a Nonlinear System?

\[ \ddot{x} + x + x^3 = A \sin \omega t \]

\[ x(t) = B \sin \omega t \]

\[ -\omega^2 B \sin \omega t + B \sin \omega t + B^3 \sin^3 \omega t = A \sin \omega t \]

\[ \sin^3 \omega t = \frac{(3 \sin \omega t - \sin 3\omega t)}{4} \]

**OPTION 1: EXACT**

\[ x(t) = B \sin \omega t + C \sin 3\omega t \]

Solution: infinite series of harmonics

**OPTION 2: APPROXIMATION**

\[ -\omega^2 B \sin \omega t + B \sin \omega t + \frac{3}{4} B^3 \sin \omega t = A \sin \omega t \]

Solve a third order polynomial

Nonlinear relation between B and A
Key Feature of Nonlinear Systems: Bifurcations

\[ \ddot{x} + 0.05\dot{x} + x + x^3 = F \sin(\omega t) \]

Displ. \( x \) (m)

1 solution  3 solutions  1 solution

Pulsation \( \omega \) (rad/s)
FRF Peak Skewness Perturbs Linear Modal Analysis

Introduction of two stable poles around 1 nonlinear mode.

B. Peeters et al., IMAC 2011
Polymax applied to F-16
Explanation of Previous Results

\[ \ddot{x} + 0.05\dot{x} + x + x^3 = F\sin\omega t \]

1. No superp. (various F, \(\omega=1.1\))
2. Sensitivity to initial conditions (F=0.15N, \(\omega=1.6\))
3. Sensitivity to forcing (F=0.05N, \(\omega=1.22/1.24\))
Outline of the Presentation

Sources of nonlinearity (with real-life examples)

Fundamental differences between linear & nonlinear dynamics

What about nonlinear modes?
Objectives of This Third Part

*Can we extend modal analysis to nonlinear systems?*

1. What do we mean by a nonlinear normal mode (NNM)?

2. How do we compute NNMs from computational models?

3. How do we extract NNMs from experimental data?
Normal Modes of Conservative Systems

Linear normal mode (LNM): synchronous periodic motion.

\[ M\ddot{q}(t) + Kq(t) = 0 \]

Nonlinear normal mode (NNM): synchronous periodic motion.

\[ M\ddot{q}(t) + Kq(t) + f_{NL}\{q(t)\} = 0 \]

Lyapunov (1907)

NDOF system: existence of at least N families of periodic solutions around the equilibrium.

Rosenberg (1960s)

Nonlinear normal mode (NNM): synchronous periodic motion.
Fundamental Difference Between LNMs and NNMs

\[ \ddot{q}_1 + (2q_1 - q_2) = 0 \]
\[ \ddot{q}_2 + (2q_2 - q_1) = 0 \]

\[ q_{1,2} = A, B \cos \omega t \]

\[ A = B, \quad \omega_1 = 1 \text{ rad/s} \]
\[ A = -B, \quad \omega_2 = \sqrt{3} \text{ rad/s} \]

LNMs

\[ \ddot{q}_1 + (2q_1 - q_2) + 0.5q_1^3 = 0 \]
\[ \ddot{q}_2 + (2q_2 - q_1) = 0 \]

\[ q_{1,2} \cong A, B \cos \omega t \]

\[ A = \pm \sqrt{\frac{8(\omega^2 - 2)(\omega^2 - 1)}{3(\omega^2 - 2)}} \]
\[ B = \frac{A}{2 - \omega^2} \]
\[ \omega_1 \in [1, \sqrt{2}] \text{ rad/s} \]
\[ \omega_2 \in [\sqrt{3}, +\infty] \text{ rad/s} \]

NNMs are frequency-amplitude dependent!
Initial conditions on displacements

Blue mass
Orange mass

(cubic)
It Is Necessary to Extend the Definition of an NNM

\[ M\ddot{q}(t) + Kq(t) = 0 \]

\[ M\ddot{q}(t) + Kq(t) + f_{NL}\{q(t)\} = 0 \]

LNM: synchronous periodic motion.

NNM: synchronous periodic motion.

Rosenberg (1960s)

Modal interactions

NNM: periodic motion (nonnecessarily synchronous).
A Frequency-Energy Plot Is a Convenient Depiction

2DOF linear

Frequency (Hz)

Energy (J)
A Frequency-Energy Plot Is a Convenient Depiction

Frequency (Hz)

Energy (J)

2DOF nonlinear

Out-of-phase NNM

In-phase NNM
LNMs and NNMs Have a Clear Conceptual Relation

But … ▶ Frequency-energy dependence
▶ Harmonics: modal interactions
▶ Bifurcations: #NNMs > #DOFs
▶ Bifurcations: stable/unstable modes
Practical Significance for Damped-Forced Responses

Linear: resonances occur in neighborhoods of LNMs.

Nonlinear: resonances occur in neighborhoods of NNMs.
## In Summary

### Clear physical meaning

<table>
<thead>
<tr>
<th>LNMs</th>
<th>NNMs</th>
</tr>
</thead>
<tbody>
<tr>
<td>YES</td>
<td>YES</td>
</tr>
</tbody>
</table>

- Structural deformation at resonance
- Synchronous vibration of the structure

### Important mathematical properties

<table>
<thead>
<tr>
<th>LNMs</th>
<th>NNMs</th>
</tr>
</thead>
<tbody>
<tr>
<td>YES</td>
<td>NO</td>
</tr>
<tr>
<td>YES</td>
<td>NO</td>
</tr>
<tr>
<td>YES</td>
<td>YES</td>
</tr>
</tbody>
</table>
Practical Computation of NNMs

Complex finite element model

Frequency-energy plot
Shooting and Pseudo-Arclength Continuation

**STEP 1:** compute an isolated periodic solution

**STEP 2:** compute the complete branch

Numerical integration (Newmark)

Newton-Raphson

Correction to the prediction

Prediction tangent to the branch
Shooting

\[ H(z_{p0}, T) \equiv z_p(T, z_{p0}) - z_{p0} = 0 \]

Periodicity condition
(2-point BVP)

Numerical solution through iterations:

\[
H\left(z_{p0}^{(0)}, T^{(0)}\right) + \frac{\partial H}{\partial z_{p0}}\bigg|_{(z_{p0}^{(0)}, T^{(0)})} \Delta z_{p0}^{(0)} + \frac{\partial H}{\partial T}\bigg|_{(z_{p0}^{(0)}, T^{(0)})} \Delta T^{(0)} + H \otimes T = 0
\]

\[
\frac{\partial H}{\partial z_0} (z_0, T) = \frac{\partial z(t, z_0)}{\partial z_0} \bigg|_{t=T} - I
\]

2n x 2n — Monodromy matrix

\[
\frac{\partial H}{\partial T} (z_0, T) = \frac{\partial z(t, z_0)}{\partial t} \bigg|_{t=T}
= g(z(T, z_0))
\]

2n x 1
Numerical Demonstration in Matlab
In-Phase and Out-of-Phase NNMs Connected

3:1 modal interaction

Frequency (Hz)

Energy (J)

Out-of-phase NNM

In-phase NNM
Neither Abstract Art Nor a New Alphabet
MS760 Aircraft (ONERA)

Bolted connections between external fuel tank and wing tip

Front connection

Rear connection
Finite Element and Reduced-Order Modeling

Finite element model (2D shells and beams, 85000 DOFs)

Condensation of the linear components of the model

Craig-Bampton technique

8 remaining nodes + 500 internal modes

Reduced model accurate in [0-100] Hz, 548 DOFs
A Close Look at Two Modes

<table>
<thead>
<tr>
<th>Mode</th>
<th>Freq. (Hz)</th>
<th>Mode</th>
<th>Freq. (Hz)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.0936</td>
<td>13</td>
<td>21.2193</td>
</tr>
<tr>
<td>2</td>
<td>0.7260</td>
<td>14</td>
<td>22.7619</td>
</tr>
<tr>
<td>3</td>
<td>0.9606</td>
<td>15</td>
<td>23.6525</td>
</tr>
<tr>
<td>4</td>
<td>1.2118</td>
<td>16</td>
<td>25.8667</td>
</tr>
<tr>
<td>5</td>
<td>1.2153</td>
<td>17</td>
<td>28.2679</td>
</tr>
<tr>
<td>6</td>
<td>1.7951</td>
<td>18</td>
<td>29.3309</td>
</tr>
<tr>
<td>7</td>
<td>2.1072</td>
<td>19</td>
<td>31.0847</td>
</tr>
<tr>
<td>8</td>
<td>2.5157</td>
<td>20</td>
<td>34.9151</td>
</tr>
<tr>
<td>9</td>
<td>3.5736</td>
<td>21</td>
<td>39.5169</td>
</tr>
<tr>
<td>10</td>
<td>8.1913</td>
<td>22</td>
<td>40.8516</td>
</tr>
<tr>
<td>11</td>
<td>9.8644</td>
<td>23</td>
<td>47.3547</td>
</tr>
<tr>
<td>12</td>
<td>16.1790</td>
<td>24</td>
<td>52.1404</td>
</tr>
</tbody>
</table>

Wing bending

Wing torsion (symmetric)
Wing Bending Mode not Affected by Nonlinearity

MAC = 1.00

MAC = 0.99
Wing Torsion Mode Is Affected by Nonlinearity

- Decrease of the natural frequency
- Internal resonances
- Mode shape slightly affected

Frequency (Hz)

MAC = 1.00

MAC = 0.98

Energy (J)
6\textsuperscript{th} NNM of the Spacecraft

![Graph showing frequency vs. energy with notable interactions at 3:1, 9:1, 26:1, and 2:1 with NNM12.](image)
6\textsuperscript{th} NNM: From a Local Mode to a Global Mode

NNM6: local deformation at low energy level

NNM6: global deformation at a higher energy level (on the modal interaction branch)

The top floor vibrates two times faster!
Excitation frequency (Hz)

Acc. (m/s²)

80N: large top floor motion
20N: no top floor motion

Sine sweep

Accelerometer at top floor

80N
20N

x4

x4

x4

x4

Acc. (m/s²)

Excitation frequency (Hz)
Experimental Confirmation

No mode of the top floor in this frequency range

Base excitation (sine sweep)
Nonlinear phase separation: **NO!**

- If several modes are excited (arbitrary forcing and/or ICs), extraction of individual NNMs is not possible generally, because modal superposition is no longer valid in the nonlinear case.

- Because the NNMs attract the dynamical flow, in some cases, the motion may end up along one dominant NNM.
Experimental Identification of NNMs?

Nonlinear phase resonance: **YES !**

- Excite a single NNM
- Exploit invariance principle:
  
  *If the motion is initiated on one specific NNM, the remaining NNMs remain quiescent for all time.*

- Identify the resulting modal parameters
Two-Step Methodology

**NNM force appropriation**

*Isolating an NNM motion using harmonic excitation*

**NNM free decay**

*Inducing single-NNM free decay*

![Diagram](image-url)

**Extract the NNM**
Necessary Ingredients

1. Appropriate excitation to isolate one NNM
2. Modal indicator (are we on the NNM of interest?)
3. Invariance principle
4. Wavelet transform and NNM extraction

Extraction of NNMs and their oscillation frequencies directly from experimental data
Nonlinear Phase Lag Quadrature Criterion

A *nonlinear* structure vibrates according to an **NNM** if the degrees of freedom have a *phase lag of 90°* with respect to the excitation *(for all harmonics!)*.

\[
x_{NNM}(t) = \sum_{k=1}^{N} X_k^{NNM} \cos(k\omega t)
\]

\[
p_{NNM}(t) = -\sum_{k=1}^{N} CX_k^{NNM} k\omega \sin(k\omega t)
\]

\[
x(t) = \sum_{k} \text{Re}(Z_k e^{ik\omega t})
\]

\[
\Delta = \frac{1}{N} \sum_{k=1}^{N} \Delta_k = 1
\]

\[
\Delta_k = \frac{\text{Re}(Z_k)^T \text{Re}(Z_k)}{Z_k^T Z_k}
\]

Nonlinear mode indicator function (MIF)
Invariance Principle

If the motion is initiated on one specific NNM, the remaining NNMs remain quiescent for all time.

Turn off the shaker
Modal curves of NNMs:
- Extracted directly from the time series

Oscillation frequencies of NNMs:
- Extracted using the wavelet transform

Reconstructed FEP from experimental data:
- Compute energy and eliminate time
Beam with a geometrical nonlinearity; benchmark of the European COST Action F3.

Experimental set-up:

► 7 accelerometers along the main beam.
► One displacement sensor (laser vibrometer) at the beam end.
► One electrodynamic exciter.
► One force transducer.
Moving Toward the First Mode

Stepped sine excitation

→ excitation  → accelerations
Moving Toward the First Mode

Stepped sine excitation

\[ \text{Displ. Amplitude (m)} \]

\[ \text{Frequency (Hz)} \]

\[ 10^4 \]

\[ \rightarrow \text{excitation} \quad \rightarrow \text{accelerations} \]
Modal Indicator

\[ \Delta = \frac{1}{N} \sum_{k=1}^{N} \Delta_k = 1 \]

Nonlinear mode indicator function (MIF)
The First Mode Vibrates in Isolation

Sine excitation
$F \sin(\omega t)$

1st NNM at a specific energy level
Frequency-Energy Dependence

Turn off the excitation

- **Acc #1 (m/s²)**
  - Time (s)

- **Acc #4 (m/s²)**
  - Time (s)

- **Acc #7 (m/s²)**
  - Acc #4 (m/s²)
Modal shapes at different energy levels

Frequencies (wavelet transform)
Validation of the Methodology

Modal curves

Modal shapes

Acc #7

Acc #3

Theoretical

Experimental

Position along the beam
Validation of the Methodology

![Graph showing frequency vs. displacement amplitude with theoretical and experimental lines.](image-url)
Concluding Remarks

_Virtually all engineering structures are nonlinear_ … but they may vibrate in linear regimes of motion.

Nonlinearity is not “plug and play”:

- Highly individualistic nature of nonlinear systems.
- Even weakly nonlinear systems can exhibit complex dynamics.
- No universal method; toolbox philosophy.

Code for NNM computation available online.
Thank you for your attention.

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